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Topological quantum phase transition in an exactly solvable model of a chiral spin liquid at finite temperature

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We study the finite-temperature nature of a quantum phase transition between an Abelian and a non-Abelian topological phase in an exactly solvable model of the chiral spin liquid of Yao and Kivelson [Phys. Rev. Lett. **99**, 247203 (2007)]. As it is exactly solvable, this model can serve as a testbed for developing better measures for describing topological quantum phase transitions. We use the global flux and entanglement entropy to characterize this phase transition, and we discuss to what extent the existence of a topologically ordered ground state with non-Abelian excitations is revealed at finite temperature.

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Characterizing and detecting topological order are one of the central questions in the field of topological phases. The challenge here is that these types of quantum ground states are not associated with any local symmetry breaking. Of broader interest in the context of quantum phase transitions (QPTs) is a question of the nature of a quantum critical point when a system enters a topologically ordered phase.[1] In a conventional QPT, in which a local order parameter starts to gain an expectation value at the quantum critical point (QCP), the nature of the QCP, although a point of measure zero, is of great significance as it governs a much larger phase space often called the "quantum critical region" [see Fig. 1(a) where we denote the expectation value of an order parameter by $\langle\phi\rangle$]. One can ask if, and to what extent, an analogy holds for topological quantum phase transitions. To answer this, we need a formulation and understanding of measures of topological order at finite temperature.

Since Wen[2] coined the term "topological order" in association with the ground-state degeneracy on topologically nontrivial surfaces of chiral spin states (proposed as a model for high-temperature superconductivity[3]), such a ground-state degeneracy has been widely used as an indicator of topological order including the non-Abelian (nA) fractional quantum Hall (FQH) states.[4] The implications of such a degeneracy on fractionalization have also been discussed.[5,6] However, the extension of this indicator, which is defined at $T=0$ and not directly accessible experimentally, to a measure at finite temperature is an open question.

More recently, the concept of "topological entanglement entropy" has been gaining interest as an indicator of topological order[7] or a topological QCP.[8] The corresponding quantity at finite temperature has also been studied.[9] However, one of the issues with topological entanglement entropy is that it does not always distinguish phases with obviously different topological orders such as weak pairing (vortices follow nA statistics) and strong pairing [vortices are Abelian (A)] $p+ip$ superconductors.[10] This shows that the information about the ground state is significantly condensed upon mapping to a single entropic quantity.

We start with the observation that the chiral spin liquid (CSL) model of Yao and Kivelson[11] can provide an ideal testbed for developing a better understanding of a topological QPT analogous to the transverse field Ising chain in the study of conventional QPTs. It has the virtues of being exactly solvable and exhibiting a nontrivial QPT between nA and A phases analogous to the weak pairing and the strong pairing $p+ip$ superconductors; the honeycomb model with three-spin interaction[12] also possesses the same virtues. (We note that the spin liquid model of Greiter and Thomale[13] has a different spin interaction but the same nA phase.) We take a twofold approach. First, we employ the notion of an "expectation value" of a global flux operator introduced by Nussinov and Ortiz[14] as a finite-temperature extension of the concept of ground-state degeneracy. Second, we compare this result to what can be learned from entanglement entropy.

Model. The exactly solvable CSL model on the star lattice[11] is a variant of a spin model with topological order introduced by Kitaev[15] on the honeycomb lattice. In this variation, ground states spontaneously break time reversal symmetry and a QPT between A and nA phases is accessible through the exact solution. For brevity we employ a Majorana fermion representation of the model. We represent spin-1/2 Pauli operators σ_i^α $(\alpha=x,y,z)$ of the original spin model at each lattice site i (Ref. 11) by four species of Majorana fermions c_i and d_i^α, $\sigma_i^\alpha=ic_id_i^\alpha$, under the constraint

$$D_i \equiv c_i d_i^x d_i^y d_i^z = 1, \qquad (1)$$

so that $\sigma^x\sigma^y\sigma^z=i$ as is expected of spin-1/2 operators. In terms of these Majorana fermions, the Hamiltonian is

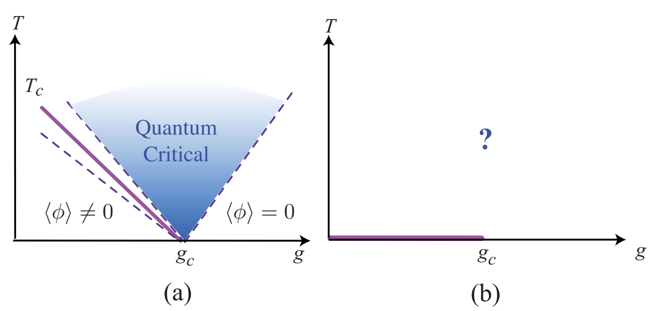

FIG. 1. (Color online) Phase diagrams in g-T phase space. (a) A typical QPT phase diagram for conventional order with the quantum fluctuations controlled by tuning parameter g. Here, we sketched a case with the dynamical critical exponent $z=1$. (b) Topological QPT phase diagram.

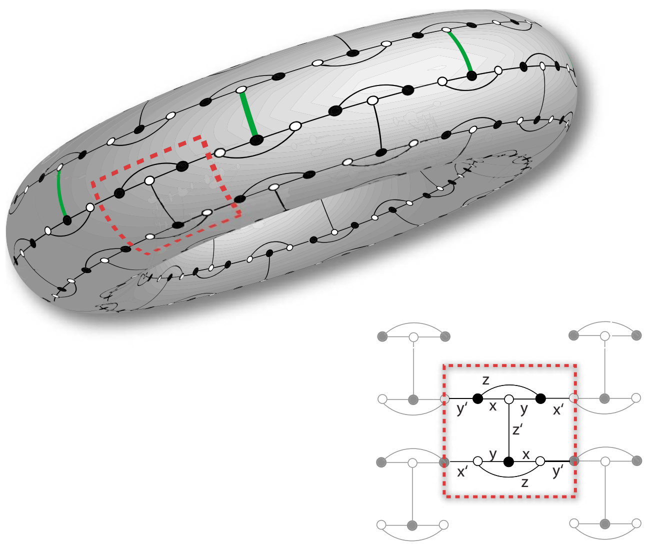

FIG. 2. (Color online) A decorated brick wall lattice, topologically equivalent to the star lattice of Ref. 11, on the surface of a torus. Solid green (solid dark gray) links denote u_{ij} configurations contributing to a global flux threading. The inset defines α links for "triangles" and α' links connecting triangles ($\alpha = x, y, z$). This labeling of links specifies components of spins interacting across the links in the original spin model.

$$\mathcal{H}[\{\hat{U}_{ij}\}] = J \sum_{x,y,z-\text{link}} \hat{U}_{ij} i c_i c_j + J' \sum_{x',y',z'-\text{link}} \hat{U}_{ij} i c_i c_j, \quad (2)$$

where $\hat{U}_{ij} \equiv -i d_i^\alpha d_j^\alpha$ is defined at each α and α' bonds between sites (i,j) (see Fig. 2) and acts as a $\mathbb{Z}_2$ gauge field living on the ij bond.[16] As $\hat{U}_{ij}$ has no dynamics ($[\hat{U}_{ij}, \mathcal{H}] = 0$) it can be replaced with a set of $\mathbb{Z}_2$ variables $u_{ij} = \pm 1$ reducing Eq. (2) to a quadratic Hamiltonian $\mathcal{H}[\{u_{ij}\}]$ parametrized by $\{u_{ij}\}$. For a loop L, the Z_2 flux is given by $\phi_L = \Pi_{ij \in L} u_{ij} = \pm 1$.

Defining $g \equiv J'/J$, $\mathcal{H}[\{u_{ij}\}]$ can be diagonalized as

$$\mathcal{H}[\{u_{ij}\}] = J \sum_{n,\vec{k}} \epsilon_{n,\vec{k}}[\{u_{ij}\}; g](b_{n,\vec{k}}^\dagger b_{n,\vec{k}} - 1/2), \quad (3)$$

by finding the complex fermion operators $b_{n,\vec{k}}$ that are linear in c_i's for momentum $\vec{k}$ and band index $n = 1,2,3$ (there are six sites per unit cell in the Majorana fermion Hamiltonian). This yields the entire spectrum $\epsilon_{n,\vec{k}}[\{u_{ij}\}; g]$. The ground states are uniform flux states with $\phi_L^0 = -1$ for all 12 plaquettes and $\phi_L^0 = 1$ or -1 for all triangular plaquettes, spontaneously breaking time reversal symmetry. A vortex on a plaquette L, defined by $\phi_L = -\phi_L^0$, costs finite energy for all g. Moreover, the uniform flux ground state is degenerate on a torus and *the topological degeneracy changes* across the nA to A QPT at $g_c = \sqrt{3}$.[11] However, care is needed for discerning physical states that satisfy the constraint (1) for each configuration of $\{u_{ij}\}$.

Topological degeneracy and the projection operator. A clue toward an extension of topological degeneracy to finite temperature lies in the g-dependent effects of the constraint (1). The constraint defines the *physical* states of the free fermion Hamiltonian (3) and is sensitive to g. Equation (1) can be implemented using a projection operator $\hat{P} = \Pi_i \frac{1}{2}(1 + D_i)$ as $D_i \hat{P} = \hat{P}$.[11,15,17] Moreover, $\hat{P}$ commutes with the original Hamiltonian $\mathcal{H}[\{\hat{U}_{ij}\}]$. We can show that whether a state survives projection only depends on the fermion parity, defined as $P_f \equiv \Pi_{ij \in x',y',z'-\text{links}} i c_i c_j$, and the parity of the number of vortex excitations[18] on triangle plaquettes.

In the uniform flux sector, all physical states have even fermion parity $P_f = 1$, which is particularly important for determining the topological degeneracy of the ground states on a torus. Topological degeneracy comes from the identical free fermion spectra, in the thermodynamic limit, in the four possible topological sectors distinguished by the choice of the $\mathbb{Z}_2$ global flux,

$$\Phi_\alpha \equiv \prod_{\langle ij \rangle \in \Gamma_\alpha} u_{ij} = \pm 1, \quad (4)$$

where $\alpha = x, y$ label two global cycles Γ_x and Γ_y.[19] Now $(\Phi_x, \Phi_y) = (\pm 1, \pm 1)$ are four distinct states where $\Phi_x = -1$ and $\Phi_y = -1$ indicate π flux threaded through the distinct holes of the torus (see Fig. 2). Normally the fermion parity in the unprojected ground-state wave function in all four topological sectors is even and independent of g. Thus, they will survive the projection and give rise to fourfold topological degeneracy. This is indeed the case on the A side ($g > g_c$). However, the fermion parity in the unprojected ground-state wave function in the $(-1,-1)$ sector on the nA side ($g < g_c$) is odd and consequently it does not survive projection.[20] Thus, there is only a threefold topological ground-state degeneracy in the nA phase. In summary, for A phases with uniform flux, all the physical states consist of an even number of fermionic quasiparticle excitations above the ground states in all four topological sectors. For nA phases with uniform flux, all physical states have an even number of fermion excitations in sectors $(1,1)$, $(1,-1)$, and $(-1,1)$, but an odd number of fermion excitations in the sector $(-1,-1)$. This has consequences not only for the topological ground-state degeneracy, but also at finite temperature as shown below.

Global flux expectation value. Motivated by the connection between the changes in the allowed physical spectrum at the topological QPT and the global fluxes, we consider the expectation value of the global flux $\langle \Phi_\alpha \rangle$ defined as[14]

$$\langle \Phi_\alpha(T) \rangle \equiv \frac{1}{\mathcal{Z}} \text{tr}\, \Phi_\alpha e^{-\mathcal{H}/T} \quad (5)$$

in a finite-size system with N sites. This ties the topological degeneracy to the spectrum and offers a natural finite-T extension of topological degeneracy. If we further restrict ourselves to uniform flux states (which is valid at $T = 0$ and is a good approximation in the vicinity of the QCP where the fermion gap is much smaller than the vortex gap), Eq. (5) can be recast as

$$\langle \Phi_\alpha \rangle = \frac{\sum_{\Phi_x, \Phi_y = \pm 1} \Phi_\alpha \mathcal{Z}^{(\Phi_x, \Phi_y)}}{\mathcal{Z}}, \quad (6)$$

where we have defined a subpartition function for each global flux sector (Φ_x, Φ_y), $\mathcal{Z}^{(\Phi_x, \Phi_y)} = \text{tr}_{(\Phi_x, \Phi_y)} \exp(-\mathcal{H}/T)|_{\text{uniform}}$,

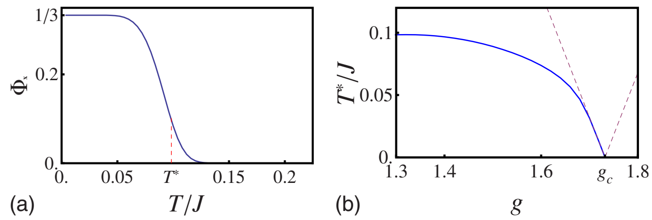

FIG. 3. (Color online) Numerical results for a 60×40 lattice. (a) The left plot shows exponential decay of $\langle\Phi_x\rangle$ for $T>T^*(g)$ at $g=1.3<g_c$. (b) The right plot shows the tuning parameter g dependence of the crossover scale T^* (solid line) and the energy gap Δ (single fermion excitation gap, dashed line) in the vicinity of g_c. Although both A and nA phases are gapped, T^* is defined only in the nA phase.

where the trace is over the states with uniform fluxes.

Clearly, in the absence of a dependence of the physical spectrum on the global flux, all the subpartition functions will be identical and $\langle\Phi_\alpha\rangle$ will be identically zero. This is the case for the A phase, and the case of toric code previously studied.[14] However, for the nA phase of the CSL model, the $(-1,-1)$ sector is projected out of the ground-state Hilbert space and hence $\mathcal{Z}^{(-1,-1)}(T=0)_{\mathrm{nA}}=0$. This yields a *finite* and definite $\langle\Phi_\alpha\rangle$ in the nA phase at $T=0$,

$$\langle\Phi_x\rangle(T=0)=\begin{cases}1/3 & (\mathrm{nA}, g<g_c)\\ 0 & (A, g>g_c),\end{cases} \tag{7}$$

The significance of Eq. (7) is that $\langle\Phi_{x/y}\rangle$ identifies a quantity that can be defined in a thermodynamic sense that *changes* at the topological QPT.

Seeking an analogy with the phase diagram in the vicinity of a conventional QCP, we turn to finite temperature. Using the definition Eq. (5), we found that there is a g-dependent crossover temperature scale $T^*(g)$ above which $\langle\Phi_x\rangle(T)$ falls off exponentially for $g<g_c$ (see the Appendix). This is demonstrated for $g=1.3$ as a representative case of $g<g_c$ in the left plot of Fig. 3. As $\langle\Phi_x\rangle\neq 0$ differentiate nA phase from A phase, T^* can be interpreted as the highest temperature at which precursor to nA ground state exists.

The crossover scale T^* is an energy scale that is distinct from the excitation gap $\Delta(g)$ (single fermion excitation gap) as demonstrated in the right plot of Fig. 3 in the sense that $T^*=0$ for $g>g_c$, while the gap is nonzero on both sides of the critical point. However, $T^*(g)$ for $g<g_c$ is proportional to $\Delta(g)$ and it has an intriguing system size dependence,

$$T^*\sim\frac{\Delta(g)}{\ln N}, \tag{8}$$

for a large system of N sites (see the Appendix for the derivation). Hence, $T^*\to 0$ as $N\to\infty$, but at a rate slower than any other quantity in the system. Therefore, the distinction between the A and the nA phases, while strictly vanishing at any finite T in the thermodynamic limit, can be meaningful in some sizable range of N. Interestingly, this system size dependence reveals similarity with the crossover scale for the finite-temperature topological entanglement entropy of the toric code in Ref. 9 and the non-Abelian $D(S_3)$ model in Ref. 21. While the results presented here are based on an exactly solvable model, the analogy to the Ising chain—where the crossover temperature has a similar size dependence—indicates that Eq. (8) should hold in other gapped non-Abelian phases. However, possible connections underlying these apparent similarities and their implications for a quantum critical region remain an open question.

Toward finite-T entanglement entropy. The topological entanglement entropy[7] of the ground-state wave function, despite its wide adoption as a measure of topological order, fails to distinguish the A phase from the nA phase.[22] The topological entanglement entropy γ is defined as the universal constant term in the entanglement entropy in addition to the usual term proportional to the perimeter of the boundary: $S_{\mathrm{ent}}=\alpha L-\gamma$. Further it is known that γ is given by the total quantum dimension of the topological phase: $\gamma=\ln\sqrt{\Sigma_\alpha d_\alpha^2}$. Since $\{d_\alpha\}=\{1,\sqrt{2},1\}$ in the nA phase while $\{d_\alpha\}=\{1,1,1,1\}$ in the A phase,[22] the topological QPT does not affect γ, which is equal to 2 in both phases.

While the zero-temperature topological entanglement entropies of the A and nA phases are identical, the different excitations of each phase will generically lead to different finite-temperature quantities. Hence, a finite-temperature extension of γ could offer a possible distinction between the two phases. An extension of the entanglement entropy to $T\neq 0$ must involve the inclusion of thermal excitations. For instance, the extension proposed in Ref. 9 (an alternative definition was proposed in Ref. 21) retains the basic form $S_{\mathrm{ent}}=-\mathrm{Tr}\,\rho_{\mathcal{A}}\ln\rho_{\mathcal{A}}$ but uses for $\rho_{\mathcal{A}}$ a thermalized reduced density matrix

$$\rho_{\mathcal{A}}(T)=\sum_\lambda\frac{e^{-E_\lambda/T}}{Z}\mathrm{Tr}_{\mathcal{B}}|\Phi_\lambda\rangle\langle\Phi_\lambda|, \tag{9}$$

where $|\Phi_\lambda\rangle$'s are energy eigenstates. Clearly Eq. (9) reduces to the usual definition at $T=0$ when only the ground state(s) enter the sum.

As a first pass through the problem, we consider the change in the entanglement entropy in the presence of a pair of vortex excitations, which are separated by the partition boundary under consideration. Figure 4 shows the result as a function of g. Explicitly, the change we are considering is defined by the difference between the entanglement entropies for a system with a pair of vortices and the ground state with no vortices, respectively. We found that there is an additional entropy of log 2 in the nA phase, which reflects the double degeneracy associated with Majorana fermion vortex core states responsible for the non-Abelian statistics of the vortices. It is clear that the characteristics of finite-energy excitations are important qualities of the topological phases.

We studied the nature of the topological QPT between a nA phase and an A phase in an exactly solvable model using finite-temperature extensions of two separate measures of topological order. The expectation value of the global flux $\langle\Phi\rangle(T)$ is a finite-temperature extension of the ground-state topological degeneracy which clearly changes at the QPT. We found that $\langle\Phi\rangle(T)$ retains the $T=0$ value for $g<g_c$ up to a crossover temperature scale which decays logarithmically with the system size. Whether this type of crossover is ubiquitous for topological phases in two spatial dimension is an

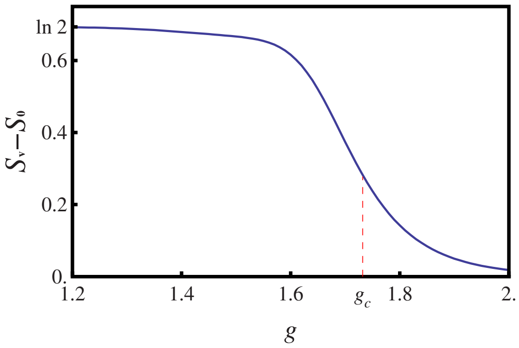

FIG. 4. (Color online) The entanglement entropy change due to a vortex pair excitation for 30×30 kagome lattice sites on a torus. The change at g_c is not sharp for this finite size, but it approaches the thermodynamic limit value of ln 2 in the nA phase.

open question. We also found an indication that the finite-temperature extension of the topological entanglement entropy $\gamma(T)$ might distinguish the nA phase from the A phase.

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APPENDIX

In the vicinity of the QPT, we can ignore the vortex excitations as the vortex gap never closes. Since $\mathcal{Z}_{CSL}^{(1,1)} = \mathcal{Z}_{CSL}^{(1,-1)} = \mathcal{Z}_{CSL}^{(-1,1)}$, Eq. (6) simplifies to

$$\langle \Phi_x \rangle(T) = [\mathcal{Z}_{CSL}^{(1,1)} - \mathcal{Z}_{CSL}^{(-1,-1)}]/[3\mathcal{Z}_{CSL}^{(1,1)} + \mathcal{Z}_{CSL}^{(-1,-1)}]. \quad \text{(A1)}$$

We now define $T^*(g)$ as the point at which $\mathcal{Z}^{(-1,-1)}(T^*)/\mathcal{Z}^{(1,1)}(T^*) = e^{-1}$.

Since in the nA phase the fermion occupation number parity is odd in the $(\Phi_x, \Phi_y) = (-1,-1)$ sector and even in all the other sectors,

$$\mathcal{Z}_{CSL}^{(1,1)} \pm \mathcal{Z}_{CSL}^{(-1,-1)} = 2\exp(-E_G/T)\prod_{n,\vec{k}}[1 \pm \exp(-\epsilon_{n,\vec{k}}/T)],$$

where E_G is the ground-state energy. We can use this to recast the global flux thermal expectation value $\langle \Phi_x \rangle(T)$,

$$\langle \Phi_x \rangle(T) = \frac{\prod_{n,\vec{k}} \tanh(\epsilon_{n,\vec{k}}/2T)}{2 + \prod_{n,\vec{k}} \tanh(\epsilon_{n,\vec{k}}/2T)}. \quad \text{(A2)}$$

Using the fact that the bandwidth does not diverge with N, we obtain Eq. (8).

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[18] After gauge fixing, the $\hat{P}$ expansion of Ref. 17 can be factorized: $\hat{P} = [1 + \Pi_i D_i]\hat{G} = [1 + P_f \Pi_{L \in \Delta,\nabla} \phi_L]\hat{G}$, where $\hat{G} = [1 + \Sigma_j D_j + \Sigma_{i<j} D_i D_j + \cdots]$ (note that $D_i^2 = 1$).

[19] This global flux operator can also be written in terms of spin variables along the global cycles (Refs. 11 and 15).

[20] The exclusion of the $(-1,-1)$ state from the ground state for $g < g_c$ is tied to the nA statistics of the vortices. A global flux threading Φ_α is equivalent to the procedure of (i) creating a vortex pair, (ii) transporting one vortex around the loop Γ_α, and (iii) annihilating the pair (Ref. 5). The $(-1,-1)$ sector is equivalent to two vortex loops linked to each other, which cannot be undone in the nA phase.

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